ANÁLISIS MATEMÁTICO VI. Curso 2006-2007

Fractales: El Conjunto de Mandelbrot

La palabra **fractal** está relacionada con la palabra *fracción*, en el sentido de parte y no todo. Fue el Prof. Benoît Mandelbrot (Varsovia, 1924) el que hace treinta años definió el conjunto de números que describen objetos en el espacio cuya dimensión en algún sentido es fraccionaria (dimensión fractal).

Mandelbrot desarrolló una geometría fractal con el objetivo de describir estructuras que no pueden ser expresados por curvas regulares conocidas de la geometría euclídea. Un ejemplo clásico de este problema es imaginar la trayectoria descrita al recorrer la costa de la isla de Tenerife tres individuos diferentes: un mototista, un hombre a pie y una hormiga siguiendo la marca del oceano en cada instante. La geometría fractal puede ser usada para modelizar matemáticamente estructuras físicas como la línea de costa anteriormente comentada, el borde de una superficie que se ha roto de forma violenta, las lineas de las nubes, etc.

El **Conjunto de Mandelbrot** es el fractal más conocido y se define formalmente en el plano complejo por medio de sucesiones. Esencialmente es el conjunto de números complejos c que definen sucesiones acotadas de la forma

$$\begin{cases} z_0 = 0 \\ z_n = z_{n-1}^2 + c \end{cases}$$

Otra familia de conjuntos fractales de interés son los **Conjuntos de Julia**. Estos conjuntos se obtienen al estudiar el comportamiento de las iteraciones de funciones holomorfas, y en particular interesan aquellos puntos donde dichas iteraciones tienen un comportamiento *caótico*. Su complementario es también de interés, aquellos puntos donde las iteraciones tienen un comportamiento *estable*, y se denomina **Conjuntos de Fatou**.

El conjunto de Mandelbrot y los de Julia tienen una estrecha relación. Si J_c es el conjunto de Julia de la función $f_c(z) = z^2 + c$, esto es, $z \in J_c$ si la sucesión

$$\begin{cases} z_0 = z \\ z_n = f(z_{n-1}) \end{cases}$$

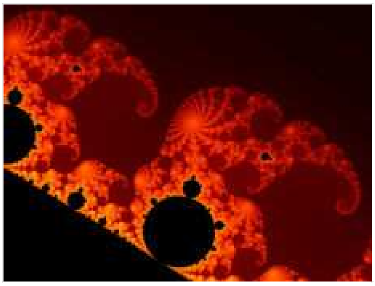
es acotada, entonces c pertenece al conjunto de Mandelbrot si, y sólo si, J_c es conexo.

Existe un gran cantidad de información en Internet y de software de generación de fractales que pueden animar al lector a familiarizarse con estos temas. Adjuntamos a continuación en pdf varias páginas web donde se puede ampliar toda esta información.

Benoît Mandelbrot

De Wikipedia, la enciclopedia libre

Benoît B. Mandelbrot (nacido en 1924) es un matemático conocido por sus trabajos sobre los fractales. Es el principal responsable del auge de este dominio de las matemáticas desde el inicio de los años ochenta, y del interés creciente del público. En efecto supo utilizar la herramienta que se estaba popularizando en ésta época - el ordenador - para trazar los más conocidos ejemplos de geometría fractal: el conjunto de Mandelbrot por supuesto, así como los conjuntos de Julia descubiertos por Gaston Julia quien inventó las matemáticas de los fractales, desarrollados luego por Mandelbrot.



Benoît Mandelbrot fue uno de los primeros científicos en utilizar los ordenadores para estudiar la fractalidad como en este ejemplo de conjunto de Mandelbrot.

Tabla de contenidos

Biografía

Nació el 20 de noviembre de 1924 en Varsovia, Polonia dentro de una familia judía culta de origen lituano. Fue introducido al mundo de las matemáticas desde pequeño gracias a sus dos tíos. Cuando su familia emigra a Francia en 1936 su tío Szolem Mandelbrot, profesor de matemáticas en el Collège de France y sucesor de Hadamardost en este puesto, toma responsabilidad de su educación. Después de realizar sus estudio en la Universidad de Lyon ingresó a la "*École Polytechnique*", a temprana edad, en 1944 bajo la dirección de Paul Lévy quien también lo influyó fuertemente. Se doctoró en matemáticas por la Universidad de París en el año 1952.

Benoît Mandelbrot durante su nombramiento como miembro de la legión de Honor.

Fue profesor de economía en la Universidad de Harvard, ingeniería en Yale, fisiología en el Colegio Albert Einstein de Medicina, y matemáticas en París y Ginebra. Desde 1958 trabajó en IBM en el Centro de Investigaciones Thomas B. Watson en Nueva York.

Logros científicos

Principal creador de la Geometría Fractal, al referirse al impacto de esta disciplina en la concepción e interpretación de los objetos que se encuentran en la naturaleza. En 1982 publicó su libro *Fractal Geometry of Nature* en el que explicaba sus investigaciones en este campo. La geometría fractal se distingue por una aproximación más abstracta a la dimensión de la que caracteriza a la geometría convencional.

Honores y premios

En 1985 recibió el premio "*Barnard Medal for Meritorious Service to Science*". En los años siguientes recibió la "*Franklin Medal*". En 1987 fue galardonado con el premio "*Alexander von Humboldt*"; también recibió la "*Medalla Steindal*" en 1988 y muchos otros premios, incluyendo la "Medalla Nevada" en 1991.

Enlaces externos

- Commons alberga contenido multimedia sobre **Benoît Mandelbrot**.
- Wikiquote alberga frases célebres de o sobre **Benoît Mandelbrot**.
- Página web B.Mandelbrot en Yale. (http://www.math.yale.edu/mandelbrot/) (en inglés)

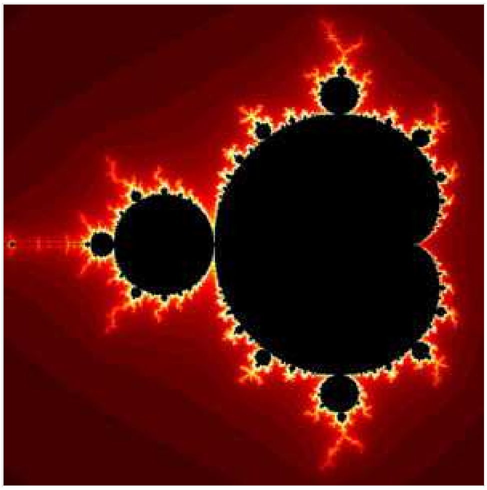
Obtenido de "http://es.wikipedia.org/wiki/Beno%C3%AEt_Mandelbrot"

Categoría: Matemáticos de Polonia

Conjunto de Mandelbrot

De Wikipedia, la enciclopedia libre

El **conjunto de Mandelbrot** es el más conocido de los conjuntos fractales, y el más estudiado.



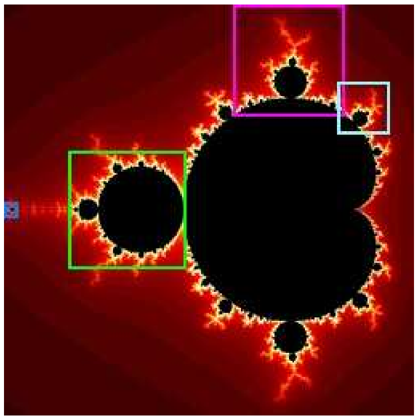
Conjunto de Mandelbrot

Este conjunto se define así, en el plano complejo:
Sea *c* un número complejo cualquiera. A partir de *c*, se construye una sucesión por inducción:

$$\begin{cases} z_0 & = & 0 & \text{(término inicial)} \\ z_{n+1} & = & z_n^2 + c & \text{(relación de inducción)} \end{cases}$$

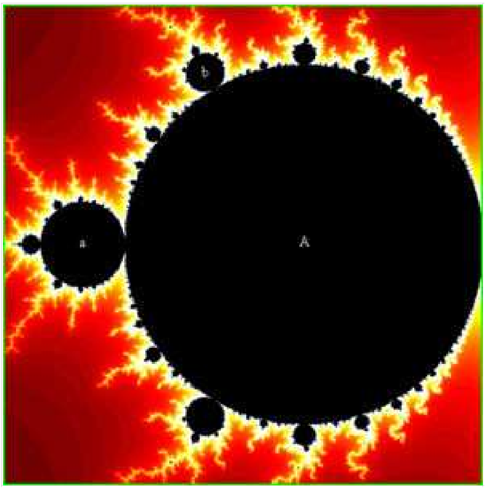
Si esta sucesión queda acotada, entonces se dice que *c* pertenece al conjunto de Mandelbrot, y si no, queda excluido del mismo. En la imagen anterior, los puntos negros pertenecen al conjunto y los de color no. Los colores dan una indicación de la velocidad con la que diverge (tiende al infinito, en módulo) la sucesión: en rojo oscuro, al cabo de pocos cálculos se sabe que el punto no está en el conjunto mientras que en blanco, se ha tardado mucho más en comprobarlo. Como no se puede calcular un sinfín de valores, es preciso poner un límite, y decidir que si los *p* primeros términos de la sucesión están acotados entonces se considera que el punto pertenece al conjunto. Al aumentar el valor de *p* se mejora la *p*recisión de la imagen.

Por otra parte, se sabe que los puntos cuya distancia al origen es superior a 2 : $x^2 + y^2 > 4$ no pertenecen al conjunto. Por lo tanto basta encontrar un solo término de la sucesión que verifique $|z_n| > 2$ para estar seguro que c no está en el conjunto.



La propiedad fundamental de los fractales es una cierta invariabilidad con relación a la escala, o dicho de otro modo, al acercarse a ciertas partes de la imagen reaparece en miniatura la imagen total. Un mismo motivo aparece a distintas escalas, a un número **infinito** de escalas.
Veámoslo más en detalle, a partir del plano siguiente (derecha):

(A partir de aquí, se puede agrandar cada imagen clicando sobre ella)



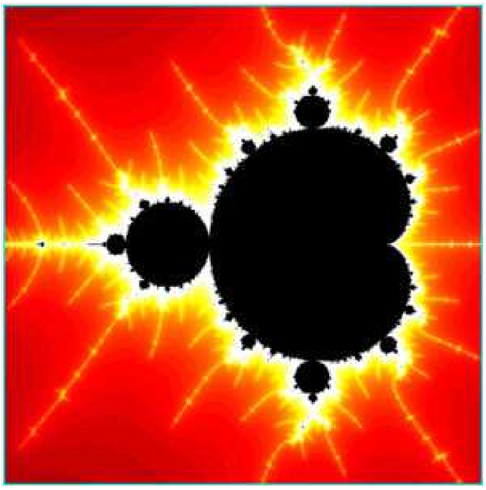
cuadro verde ampliado

Al agrandar el cuadro verde, se obtiene la imagen siguiente (izquierda):

Salta a la vista que la *bola* negra *a* es una reducción exacta de la *bola A*. La protuberancia a la izquierda de *a* también es una reducción exacta de *a*, y el proceso sigue indefinidamente.

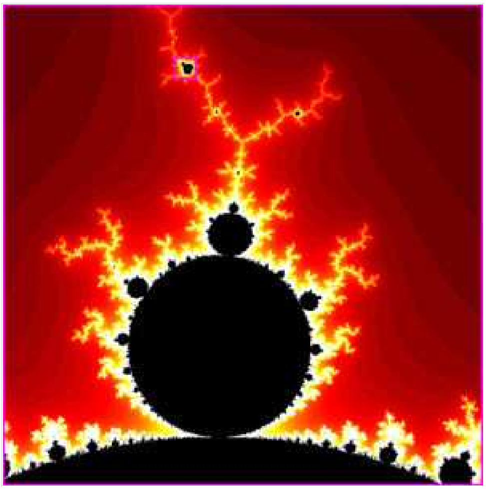
También se puede observar que la *bola b* es una reducción de *A* (una reducción combinada con una rotación, es decir que *b* se obtiene de *A* mediante una semejanza). Mirando mejor, se nota un sinfín de protuberancias *semejantes* a *A*.

Volviendo al plano, escojamos esta vez el cuadro azul oscuro a la izquierda. Al agrandarlo, obtenemos (derecha):

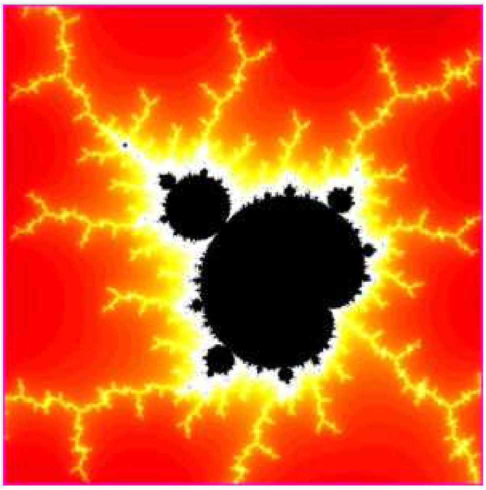

cuadro azul ampliado

Su parecido a la imagen inicial es obvio. El proceso se puede repetir un sinfín de veces, empezando por agrandar la pequeña mancha negra a la izquierda del cuadro.

Ahora, ampliemos el cuadro violeta del plano:

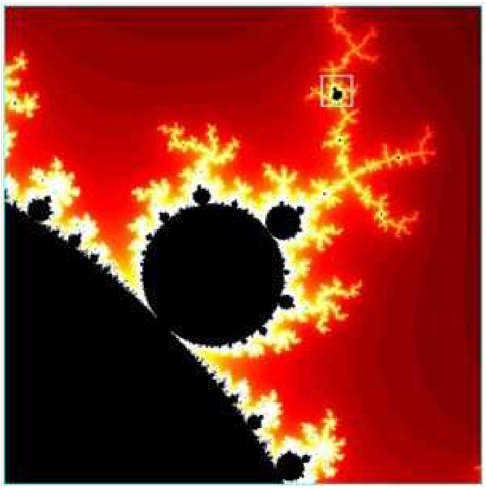

cuadro violeta ampliado

En esta imagen aparece una mancha arriba a la izquierda que tiene la misma forma que la imagen inicial. Al mirar más de cerca, se obtiene (abajo):

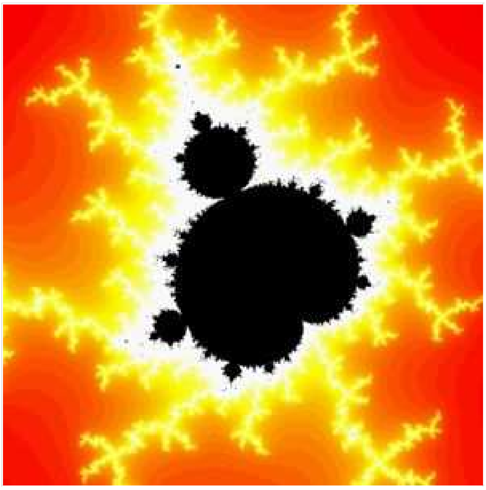

segundo cuadro violeta ampliado

Y una vez más, el parecido salta a la vista.

Ahora, agrandemos el cuadro azul claro de la derecha del plano:


cuadro azul claro ampliado

Acerquémonos al cuadro blanco de la última imagen:


cuadro blanco ampliado

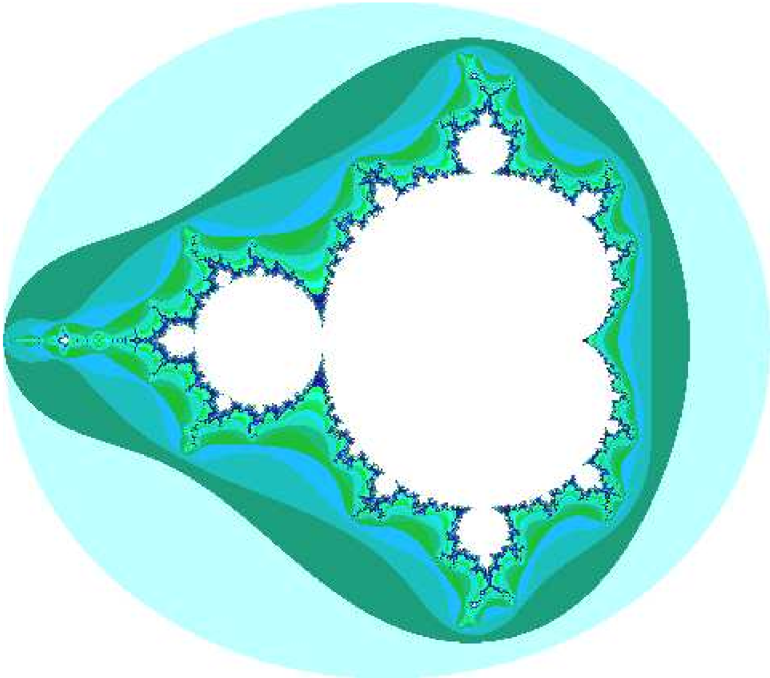
Aquí se nota una ligera deformación de la figura inicial. Sin embargo, esta imagen sigue siendo **isomorfa** a la inicial. Y claro, alrededor de cada *clon* de la forma inicial existen otros clones minúsculos, en las mismas posiciones relativas que en la figura global. El proceso no tiene fin.

Existe otra manera de definir este conjunto:

Es el conjunto de los complejos *c* para los que el conjunto de Julia asociado a $f_c(z) = z^2 + c$ es conexo.

Otra representación

En esta imagen, el conjunto es, naturalmente, el mismo, pero las *líneas de nivel* (que separan los colores, fuera del conjunto) no son idénticas. Esto se debe a que no se ha empleado el mismo criterio de divergencia: en esta imagen es realmente $|z_n| > 2$, mientras que en las anteriores era $|z_n| > 10$, por razones estéticas, ya que así se obtiene una imagen inicial menos oscura.



Enlaces externos

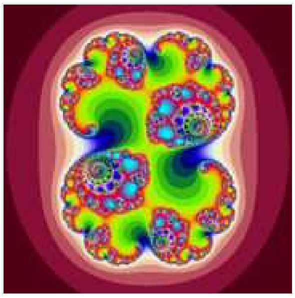
-  Commons alberga contenido multimedia sobre **Conjunto de Mandelbrot**.

Categorías: EL | Fractales

Conjunto de Julia

De Wikipedia, la enciclopedia libre

Los **conjuntos de Julia** (así llamados por el matemático Gaston Julia) son una familia de conjuntos fractales, que se obtienen al estudiar el comportamiento de los números complejos al ser iterados por una función holomorfa.


Conjunto de Julia, un fractal. C = [0.285, 0.01]

El conjunto de Julia de una función holomorfa *f* esta constituido por aquellos puntos que bajo la iteración de *f* tienen un comportamiento 'caótico'. El conjunto se denota $J(f)$.

En el otro extremo se encuentra el conjunto de Fatou (en honor del matemático Pierre Fatou), que consiste de los puntos que tienen un comportamiento 'estable' al ser iterados. El conjunto de Fatou de una función holomorfa *f* se denota $F(f)$ y es el complemento de $J(f)$.

Polinomios cuadráticos

Una familia muy notable de conjuntos de Julia se obtienen a partir de funciones cuadráticas simples: $f_c(z) = z^2 + c$, donde c es un número complejo. El conjunto de Julia que se obtiene a partir de esta función se denota J_c.

Un algoritmo para obtener el conjunto de Julia de $f_c(z) = z^2 + c$ es el siguiente:

Para todo complejo z se construye por la siguiente sucesión:

$$z_0 = z$$
$$z_{n+1} = z_n^2 + c$$

Si esta sucesión queda acotada, entonces se dice que z pertenece al conjunto de Julia de parámetro c, denotado por J_c; de lo contrario, z queda excluido de éste.

En las imágenes anteriores, los puntos negros pertenecen al conjunto y los de color no. Los colores dan una indicación de la velocidad con la que diverge la sucesión (su módulo tiende a infinito): en rojo oscuro, al cabo de pocos cálculos se sabe que el punto no está en el conjunto; y en blanco, se ha tardado mucho más en comprobarlo. Como no se pueden calcular infinitos valores, es preciso poner un límite, y decidir que si los n primeros términos de la sucesión están acotados, el punto pertenece al conjunto. Al aumentar el valor de n se mejora la precisión de la imagen.


Conjunto de Julia en 3D

Se puede demostrar que si $|z_n| > 2$ entonces la sucesión diverge y el punto z no pertenecen al conjunto de Julia. Por lo tanto, basta encontrar un solo término de la sucesión que verifique $|z_n| > 2$ para tener la certeza de que z no está en el conjunto.

Existe una relación muy fuerte entre los conjuntos de Julia y el conjunto de Mandelbrot denotado por M, debido a la similitud de sus definiciones:

Se dice que c pertenece a M si y sólo si J_c es conexo.

Los resultados más vistosos se obtienen al tomar el parámetro c en la frontera de M, pues si c esta en el interior de M resulta que J_c toma el aspecto de un objeto redondo, poco fractal, y sólo el borde tiene la apariencia de fractal. Por ejemplo si $c = 0$ resulta que el conjunto de Julia es la circunferencia unitaria, con centro en el origen de coordenadas.

En las imágenes, se han tomado como valores de c: -1,3 + 0,00525·i ; -0,72 – 0,196·i ; -0,1 + 0,87i y -0,51 – 0,601·i, por razones estéticas.

Se pueden generalizar estos conjuntos tomando otras relaciones de inducción: $z_{n+1} = f(z_n)$ con cualquier función compleja f. Se puede también generalizar a cualquier dimensión, y emplear varias funciones en lugar de una sola.

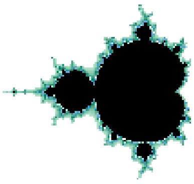
Enlaces externos

- Commons alberga contenido multimedia sobre **Conjunto de Julia**.

Obtenido de "http://es.wikipedia.org/wiki/Conjunto_de_Julia"

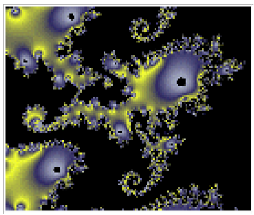
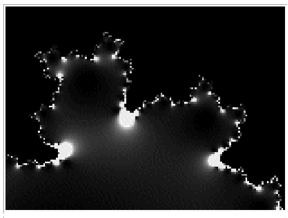
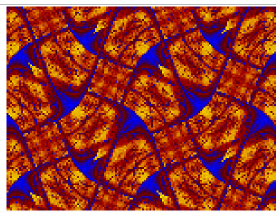
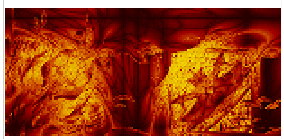
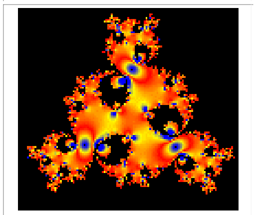
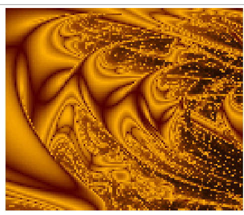
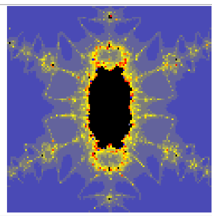
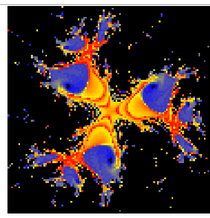
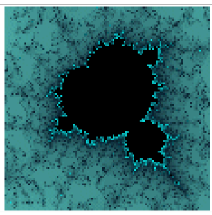
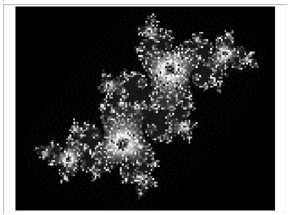
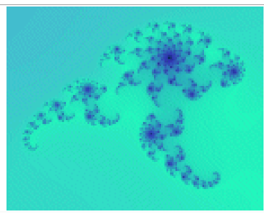
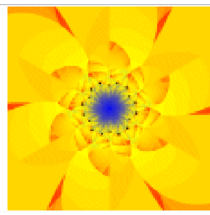
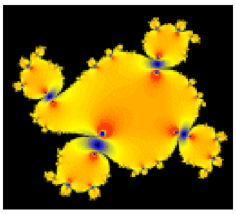
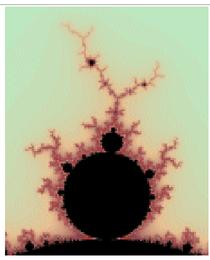
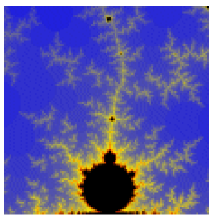
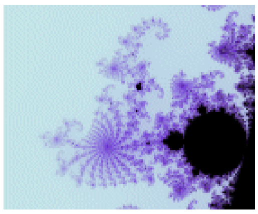
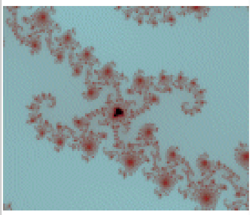
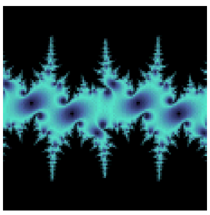
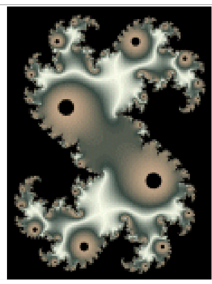
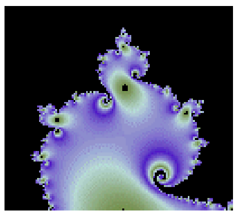
Categoría: Fractales

Fractales - π



Esta sala de la galería de **Epsilones** está íntegramente dedicada a las imágenes fractales. Cada pequeña imagen, que puedes ampliar con un simple *clic* de ratón, está acompañada del tamaño de la ampliación en kilobytes y de un número que remite a una tabla donde aparecen algunas de las características de la imagen.

Si lo que quieres es crear tus propias imágenes, echa un vistazo al laboratorio: allí encontrarás un programa para la Exploración del conjunto de Mandelbrot.

 25 - 73 Kb	 26 - 27 Kb	 27 - 156 Kb	 28 - 158 Kb
 21 - 57 Kb	 22 - 143 Kb	 23 - 41 Kb	 24 - 64 Kb
 17 - 132 Kb	 18 - 55 Kb	 19 - 65 Kb	 20 - 45 Kb
 13 - 40 Kb	 14 - 50 Kb	 15 - 93 Kb	 16 - 88 Kb
			

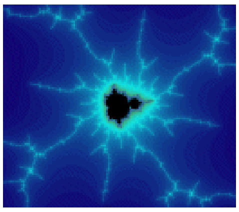
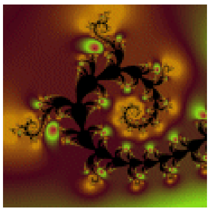
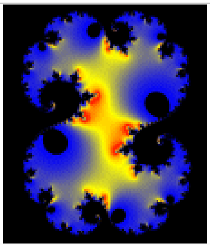
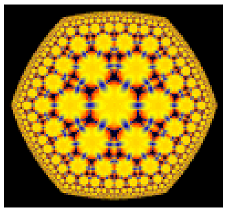
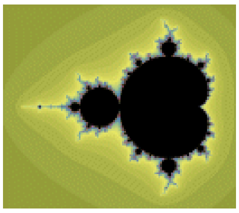
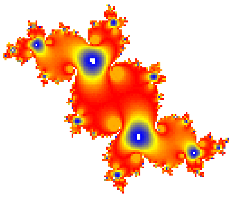
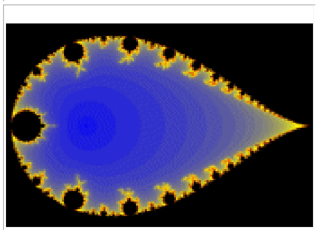
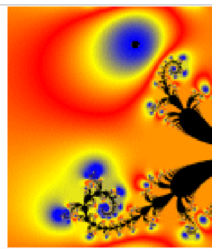
9 - 92 Kb	10 - 50 Kb	11 - 59 Kb	12 - 48 Kb
			
5 - 68 Kb	6 - 80 Kb	7 - 49 Kb	8 - 111 Kb
			
1 - 38 Kb	2 - 45 Kb	3 - 30 Kb	4 - 87 Kb

Imagen	Tipo	Modo	Ecuación	Otras características	*Software*
1	Conjunto de Mandelbrot	Divergencia	$z^2 + c$		
2	Conjunto de Julia	Convergencia	$z^2 + c$		
3	Conjunto de Mandelbrot	Inversión	$z^2 + c$		
4	Conjunto de Julia	Convergencia	$c \cdot e^x$	Ampliación	
5	Conjunto de Mandelbrot	Divergencia	$z^2 + c$	Ampliación	
6	Conjunto de Julia	Convergencia	$c \cdot e^x$	Ampliación	
7	Conjunto de Julia	Convergencia	$z^2 + c$		
8	Conjunto de Julia	Convergencia	$z^{3/2} + c$		
9	Conjunto de Mandelbrot	Divergencia	$z^2 + c$	Ampliación	
10	Conjunto de Julia	Convergencia	$\cos z + c$		
11	Conjunto de Julia	Convergencia	$z^2 + c$		
12	Conjunto de Julia	Convergencia	$z^2 + c$	Ampliación	
13	Conjunto de Julia	Convergencia	$z^4 + z^3 + c$		
14	Conjunto de Mandelbrot	Divergencia	$z^2 + c$	Ampliación	
15	Conjunto de Mandelbrot	Divergencia	$\cos z + c$	Ampliación	
16	Conjunto de Mandelbrot	Divergencia	$z^2 + c$	Ampliación	
17	Conjunto de Mandelbrot	Divergencia	$z^2 + c$	Ampliación	
18	Conjunto de Julia	Divergencia	$z^2 + c$		
19	Conjunto de Julia	Divergencia	$z^2 + c$	Ampliación	
20	Conjunto de Julia	Inversión	$z^{3/2} + c$		
21	Conjunto de Julia	Convergencia	$z^3 + c$		
22	Diagrama de Lyapunov		$r \cdot x \cdot (1-x)$	$B^7A^2B^9(BA)^9A^7B^2A^7$	

23	Conjunto de Mandelbrot	Divergencia	$z^3 + c$	Cambio: $b_{n+1} = 2 \cdot \text{sen}(\text{Im}(z^3)) + c_2$ (Nano)	Nano
24	Conjunto de Julia	Convergencia	$z^3 + c$	Cambio: $a_{n+1} = 2 \cdot \text{sen}(\text{Re}(z^3)) + c_1$ (Nano)	
25	Conjunto de Mandelbrot	Convergencia	$z^2 + c$	Ampliación	
26	Conjunto de Mandelbrot	Convergencia	$z^2 + c$	Ampliación	
27	Diagrama de Lyapunov		$r \cdot \text{sen}^2(x+r)$	AB	
28	Diagrama de Lyapunov		$r \cdot \text{sen}^2(x+r)$	AAAAAABBBBBB	

► inicio de la página

Epsilones
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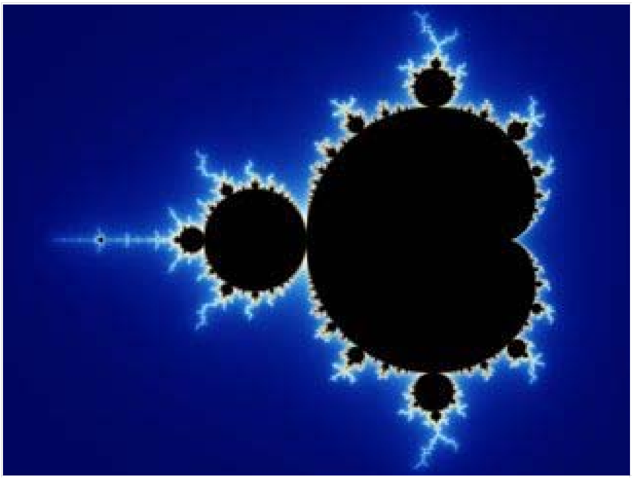
Mandelbrot set

From Wikipedia, the free encyclopedia

The **Mandelbrot set** is a fractal that has become popular outside of mathematics both for its aesthetic appeal and its complicated structure, arising from a simple definition. This is largely due to the efforts of Benoît Mandelbrot and others, who worked hard to communicate this area of mathematics to the general public.



Initial image of a Mandelbrot set zoom sequence with continuously colored environment.

Contents

History

The Mandelbrot set has its place in complex dynamics, a field first investigated by the French mathematicians Pierre Fatou and Gaston Julia
at the beginning of the 20th century. The first pictures of it were drawn in 1978 by Brooks and Matelski as part of a study of Kleinian Groups.[1]

Mandelbrot studied the parameter space of quadratic polynomials in an article which appeared in 1980.[2] The mathematical study of the Mandelbrot set really began with work by the mathematicians Adrien Douady and

John H. Hubbard,[3] who established many fundamental properties of *M*, and named the set in honor of Mandelbrot.

The work of Douady and Hubbard coincided with a huge increase in interest in complex dynamics, and the study of the Mandelbrot set has been a centerpiece of this field ever since. It would be futile to attempt to make a list of all the mathematicians who have contributed to our understanding of this set since then, but such a list would certainly include Mikhail Lyubich, Curt McMullen, John Milnor, Mitsuhiro Shishikura and Jean-Christophe Yoccoz.

Formal definition

The Mandelbrot set *M* is defined by a family of complex quadratic polynomials

$$f_c : \mathbb{C} \to \mathbb{C}$$

given by

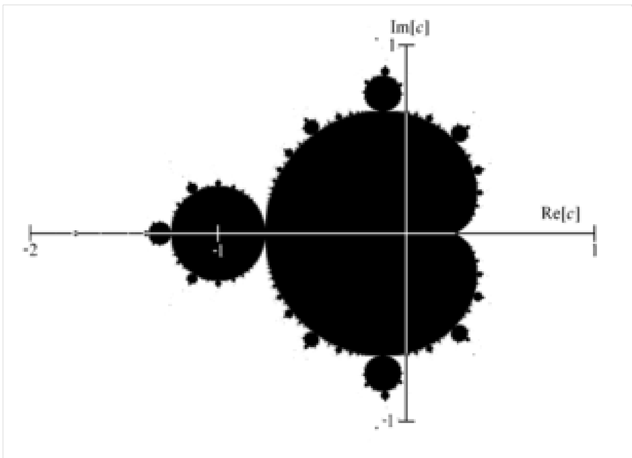
$$f_c(z) = z^2 + c.$$

where *c* is a complex parameter. For each *c*, one considers the behaviour of the sequence $(0, f_c(0), f_c(f_c(0)), f_c(f_c(f_c(0))), \ldots)$ obtained by iterating $f_c(z)$ starting at $z = 0$, which either escapes to infinity or stays within a disk of some finite radius. The Mandelbrot set is defined as the set of all points *c* such that the above sequence does *not* escape to infinity.

More formally, if $f_c^n(z)$ denotes the *n*th iterate of $f_c(z)$ (i.e. $f_c(z)$ composed with itself *n* times) the Mandelbrot set is the subset of the complex plane given by

$$M = \left\{ c \in \mathbb{C} : \sup_{n \in \mathbb{N}} |f_c^n(0)| < \infty \right\}.$$

Mathematically, the Mandelbrot set is just a set of complex numbers. A given complex number *c* either belongs to *M* or it does not. A picture of the Mandelbrot set can be made by coloring all the points *c* which belong to *M* black, and all other points white. The more colorful pictures usually seen are generated by coloring points not in the set according to how quickly or slowly the sequence $|f_c^n(0)|$ diverges to infinity. See the section on computer drawings below for more details.



A mathematician's depiction of the Mandelbrot set *M*. A point *c* is colored black if it belongs to the set, and white otherwise.

The Mandelbrot set can also be defined as the connectedness locus of the family of polynomials $f_c(z)$. That is, it is the subset of the complex plane consisting of those parameters *c* for which the Julia set of f_c is connected.

Basic properties

The Mandelbrot set is a compact set, contained in the closed disk of radius 2 around the origin. In fact, a point *c* belongs to the Mandelbrot set if and only if $|f_c^n(0)| \leq 2$ for all $n > 0$. In other words, if the absolute value of $f_c^n(0)$ ever becomes larger than 2, the sequence will escape to infinity.

The intersection of *M* with the real axis is precisely the interval $[-2, 0.25]$. The parameters along this interval can be put in one-to-one correspondence with those of the real logistic family,

$$z \mapsto \lambda z(z-1), \quad \lambda \in [1,4].$$

The correspondence is given by

$$c = \frac{1-(\lambda-1)^2}{4}.$$

In fact, this gives a correspondence between the entire parameter space of the logistic family and that of the Mandelbrot set.

The area of the Mandelbrot set is estimated to be 1.506 591 77 ± 0.000 000 08. This is conjectured to be $\sqrt{6\pi - 1} - e$ = 1.506591651… exactly. [1] (http://www.mrob.com/pub/muency/pixelcounting.html)

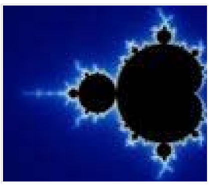
Douady and Hubbard have shown that the Mandelbrot set is connected. In fact, they constructed an explicit conformal isomorphism between the complement of the Mandelbrot set and the complement of the closed unit disk. Mandelbrot had originally conjectured that the Mandelbrot set is disconnected. This conjecture was based on computer pictures generated by programs which are unable to detect the thin filaments connecting different parts of *M*. Upon further experiments, he revised his conjecture, deciding that *M* should be connected.

The dynamical formula for the uniformisation
of the complement of the Mandelbrot set, arising from Douady and Hubbard's proof of the connectedness of *M*, gives rise to external rays
of the Mandelbrot set. These rays can be used to study the Mandelbrot set in combinatorial terms, and form the backbone of the Yoccoz parapuzzle.

The boundary of the Mandelbrot set is exactly the bifurcation locus of the quadratic family; that is, the set of parameters *c* for which the dynamics changes abruptly under small changes of *c*. It can be constructed as the limit set of a sequence of plane algebraic curves, the *Mandelbrot curves*, of the general type known as polynomial lemniscates. The Mandelbrot curves are defined by setting $p_0=z$, $p_n=p_{n-1}^2+z$, and then interpreting the set of points $|p_n(z)|=1$ in the complex plane as a curve in the real Cartesian plane of degree 2^{n+1} in x and y.

Image gallery of a zoom sequence

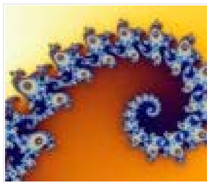
The following example of an image sequence zooming to a selected *c* value gives an impression of the infinite richness of different geometrical structures and explains some of their typical rules. The magnification of the last image relative to the first one is about 60,000,000,000 to 1. Relating to an ordinary monitor it represents a section of a Mandelbrot set with a diameter of 20 million kilometres. Its border would show an inconceivable amount of different fractal structures.


Start


Step 1


Step 2


Step 3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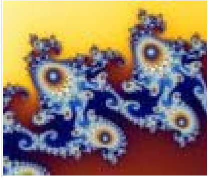
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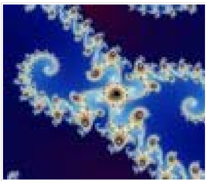
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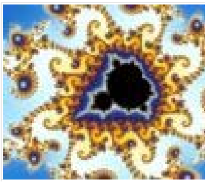
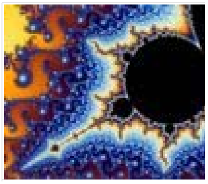
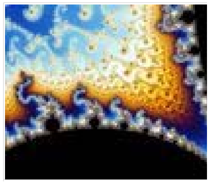
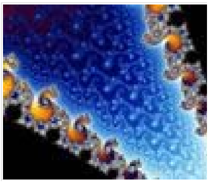
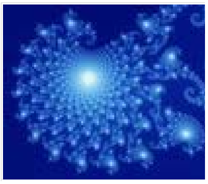

Step 7

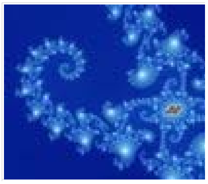

Step 8

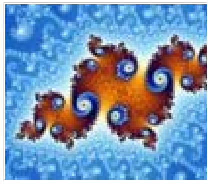

Step 9


Step 10


Step 11

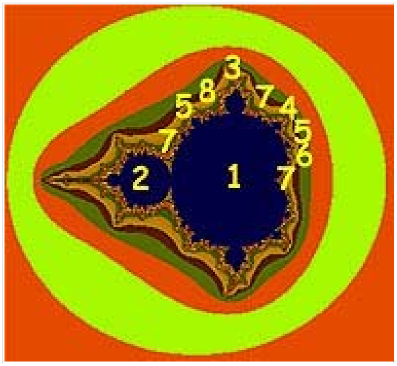

Step 12


Step 13


Step 14

- Start: Mandelbrot set with continuously colored environment.
- Step 1: Gap between the "head" and the "body" also called the "seahorse valley".
- Step 2: On the left double-spirals, on the right "seahorses".
- Step 3: "Seahorse" upside down. Its "body" is composed by 25 "spokes" consisting of 2 groups of 12 "spokes" each and one "spoke" connecting to the main cardioid. These 2 groups can be attributed by some kind of metamorphosis to the 2 "fingers" of the "upper hand" of the Mandelbrot set. Therefore the number of "spokes" increases from one "seahorse" to the next by 2. The "hub" is a so called Misiurewicz point. Between the "upper part of the body" and the "tail" a distorted small copy of the Mandelbrot set called satellite can be recognized.
- Step 4: The central endpoint of the "seahorse tail" is also a Misiurewicz point.
- Step 5: Part of the "tail". There is only one path consisting of the thin structures which leads through the whole "tail". This zigzag path passes the "hubs" of the large objects with 25 "spokes" at the inner and outer border of the "tail". It makes sure, that the Mandelbrot set is a so called simply connected set. That means there are no islands and no loop roads around a hole.
- Step 6: Satellite. The two "seahorse tails" are the beginning of a series of concentric crowns with the satellite in the center.
- Step 7: Each of this crowns consists of similar "seahorse tails". Their number increases with powers of 2, a typical phenomenon in the environment of satellites. The unique path to the spiral center mentioned in zoom step 5 passes the satellite from the groove of the cardioid to the top of the "antenna" on the "head".
- Step 8: "Antenna" of the satellite. Several satellites of second order can be recognized.
- Step 9: The "seahorse valley" of the satellite. All the structures from the image of zoom step 1 reappear.
- Step 10: Double-spirals and "seahorses". Unlike the image of zoom step 2 they have appendices consisting of structures like "seahorse tails". This demonstrates the typical linking of n+1 different structures in the environment of satellites of the order n, here for the simplest case n=1.
- Step 11: Double-spirals with satellites of second order. Analog to the "seahorses" the double-spirals can be interpreted as a metamorphosis of the "antenna".
- Step 12: In the outer part of the appendices islands of structures can be recognized. They have a shape like Julia sets J_c. The largest of them can be found in the center of the "double-hook" on the right side.
- Step 13: Part of the "double-hook".
- Step 14: On the first sight these islands seem to consist of infinitely many parts like Cantor sets, as it is actually the case for the corresponding Julia set J_c. Here they are connected by tiny structures so that the whole represents a simply connected set. These tiny structures meet each other at a satellite in the center which is too small to be recognized at this magnification. The value of c for the corresponding J_c is not that of the image center but has relative to the main body of the Mandelbrot set the same position as the center of this image relative to the satellite shown in zoom step 7.

The main cardioid and period bulbs



Periods of hyperbolic components

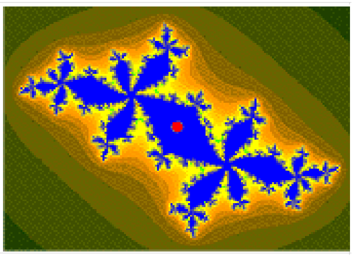
Upon looking at a picture of the Mandelbrot set, one immediately notices the large cardioid-shaped region in the center. This *main cardioid* is the region of parameters c for which f_c has an attracting fixed point. It consists of all parameters of the form

$$c = \frac{1-(\mu-1)^2}{4}$$

for some μ in the open unit disk.

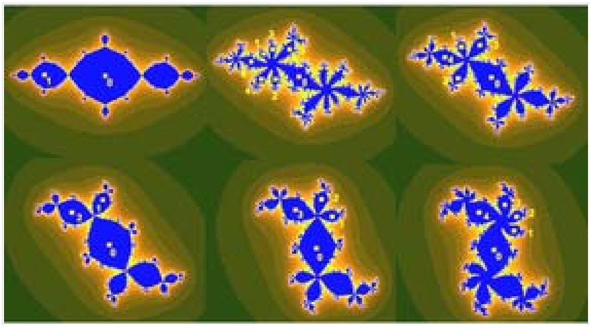
To the left of the main cardioid, attached to it at the point $c = -3/4$, a circular-shaped bulb is visible. This bulb consists of those parameters c for which f_c has an attracting cycle of period 2. This set of parameters is an actual circle, namely that of radius 1/4 around -1.

There are infinitely many other bulbs attached to the main cardioid: for every rational number $\frac{p}{q}$, with p and q coprime, there is such a bulb attached at the parameter : $c_{\frac{p}{q}} = \frac{1-\left(e^{2\pi i\frac{p}{q}}-1\right)^2}{4}$.


Attracting cycle in 2/5-bulb plotted over Julia set (animation)

This bulb is called the $\frac{p}{q}$-*bulb* of the Mandelbrot set. It consists of parameters which have an attracting cycle of period q and combinatorial rotation number $\frac{p}{q}$. More precisely, the q periodic Fatou components containing the attracting cycle all touch at a common point (commonly called the α-*fixed point*). If we label these components $U_0, \ldots, U_{q-1}$ in counterclockwise orientation, then f_c maps the component U_j to the component $U_{j+p\,(\mathrm{mod}\,q)}$.

The change of behavior occurring at $c_{\frac{p}{q}}$ is known as a bifurcation: the attracting fixed point "collides" with a repelling period *q*-cycle. As we pass through the bifurcation parameter into the $\frac{p}{q}$-bulb, the attracting fixed point turns into a repelling fixed point (the α-fixed point), and the period *q*-cycle becomes attracting.



Attracting cycles and Julia sets for parameters in the 1/2, 3/7, 2/5, 1/3, 1/4 and 1/5 bulbs

Hyperbolic components

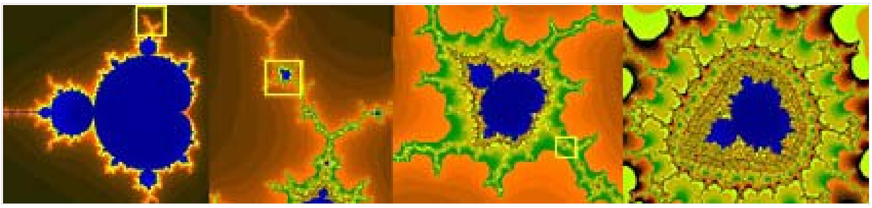
All the bulbs we encountered in the previous section were interior components of the Mandelbrot set in which the maps f_c have an attracting periodic cycle. Such components are called *hyperbolic components*.

It is conjectured that these are the *only* interior regions of *M*. This problem, known as *density of hyperbolicity*, may be the most important open problem in the field of complex dynamics. Hypothetical non-hyperbolic components of the Mandelbrot set are often referred to as "queer" components.

For *real*
quadratic polynomials, this question was answered positively in the 1990s independently by Lyubich and by Graczyk and Świątek. (Note that hyperbolic components intersecting the real axis correspond exactly to periodic windows in the Feigenbaum diagram. So this result states that such windows exist near every parameter in the diagram.)

Not every hyperbolic component can be reached by a sequence of direct bifurcations from the main cardioid of the Mandelbrot set. However, such a component *can* be reached by a sequence of direct bifurcations from the main cardioid of a little Mandelbrot copy (see below).

Little Mandelbrot copies



Little Mandelbrot copies

The Mandelbrot set is self-similar in the sense that small distorted versions of itself can be found at arbitrarily small scales near any point of the boundary of the Mandelbrot set. This phenomenon is explained by Douady and Hubbard's theory of renormalization.

Local connectivity of the Mandelbrot set

It is conjectured that the Mandelbrot set is locally connected. This famous conjecture is known as *MLC* (for *Mandelbrot Locally Connected*). By the work of Douady and Hubbard, this conjecture would result in a simple abstract "pinched disk" model of the Mandelbrot set. In particular, it would imply the important *hyperbolicity conjecture* mentioned above.

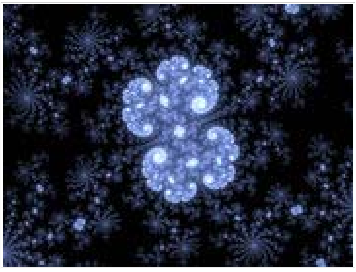
The celebrated work of Yoccoz
established local connectivity of the Mandelbrot set at all finitely-renormalizable parameters; that is, roughly speaking those which are contained only in finitely many small Mandelbrot copies. Since then, local connectivity has been proved at many other points of *M*, but the full conjecture is still open.

Further results

The Hausdorff dimension of the boundary of the Mandelbrot set equals 2 by a result of Mitsuhiro Shishikura.[4] It is not known whether the boundary of the Mandelbrot set has positive planar Lebesgue measure.

In the Blum-Shub-Smale model of real computation, the Mandelbrot set is not computable, but its complement is computably enumerable. However, many simple objects (e.g., the graph of exponentiation) are also not computable in the BBS model. At present it is unknown whether the Mandelbrot set is computable in models of real computation based on computable analysis, which correspond more closely to the intuitive notion of "plotting the set by a computer". Hertling has shown that the Mandelbrot set is computable in this model if the hyperbolicity conjecture is true.

Relationship with Julia sets


An "embedded Julia set"

As a consequence of the definition of the Mandelbrot set, there is a close correspondence between the geometry of the Mandelbrot set at a given point and the structure of the corresponding Julia set.

This principle is exploited in virtually all deep results on the Mandelbrot set. For example, Shishikura proves that, for a dense set of parameters in the boundary of the Mandelbrot set, the Julia set has Hausdorff dimension two, and then transfers this information to the parameter plane. Similarly, Yoccoz first proves the local connectivity of Julia sets, before establishing it for the Mandelbrot set at the corresponding parameters. Adrien Douady phrases this principle as

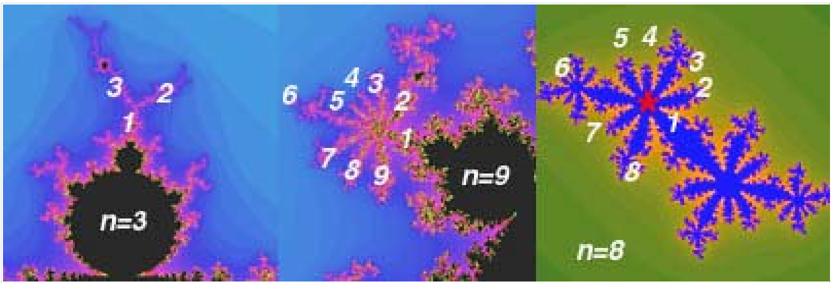
> Plough in the dynamical plane, and harvest in parameter space.

Geometry of the Mandelbrot set

Recall that, for every rational number $\frac{p}{q}$, where p and q are relatively prime, there is a hyperbolic component of period q bifurcating from the main cardioid. The part of the Mandelbrot set connected to the main cardioid at this bifurcation point is called the $\frac{p}{q}$-*limb*. Computer experiments suggest that the diameter of the limb tends to zero like $\frac{1}{q^2}$. The best current estimate known is the famous *Yoccoz-inequality*, which states that the size tends to zero like $\frac{1}{q}$.

A period q-limb will have $q - 1$ "antennas" at the top of its limb. We can thus determine the period of a given bulb by counting these antennas.

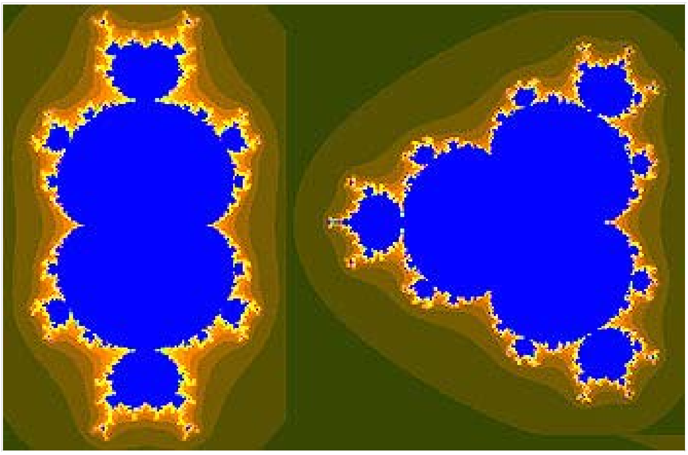
Generalizations



cycle periods and antennas

Sometimes the connectedness loci of families other than the quadratic family are also referred to as the *Mandelbrot sets* of these families.

The connectedness loci of the unicritical polynomial families $f_c = z^d + c$ for $d > 2$ are often called *Multibrot sets*.


Multibrot sets of degrees 3 and 4

For general families of holomorphic functions, the *boundary* of the Mandelbrot set generalizes to the bifurcation locus, which is a natural object to study even when the connectedness locus is not useful.

It is also possible to consider similar constructions in the study of non-analytic mappings. Of particular interest is the *tricorn*, the connectedness locus of the anti-holomorphic family

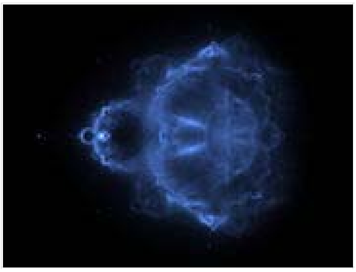
$$z \mapsto \bar{z}^2 + c.$$

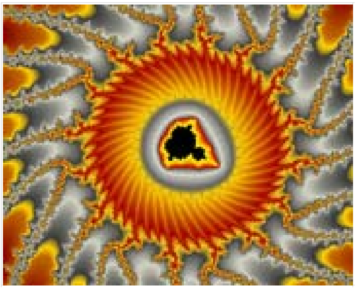
The tricorn (also sometimes called the *Mandelbar set*) was encountered by Milnor in his study of parameter slices of real cubic polynomials. It is *not* locally connected. This property is inherited by the connectedness locus of real cubic polynomials.

Computer drawings of the Mandelbrot set

Algorithms :

- parallel
- sequential
  - Escape time algorithm
    - boolean version (draws M-st and its exterior using 2 colors) = Mandelbrot algorithm
    - discrete (integer) version = level set method (LSM/M); draws Mandelbrot set and color bands in its exterior
    - continous version
    - level curves version = draws lemniscates of Mandelbrot set
    - decomposition of exterior of andelbrot set
  - Hubbard-Douady potential of Mandelbrot set (CPM/M)
  - Distance Estimation Method for Mandelbrot set = Milnor algorithm (DEM/M)
  - abstract M-set


Buddhabrot method


Still image of a movie of increasing magnification on 0.001643721971153 + 0.822467633298876i

Escape time algorithm

The simplest algorithm for generating a representation of the Mandelbrot set is known as the "escape time" algorithm. A repeating calculation is performed for each x, y point in the plot area and based on the behaviour of that calculation, a colour is chosen for that pixel.

The x and y location of each point are used as starting values in a repeating, or iterating calculation (described in detail below). The result of each iteration is used as the starting values for the next. The values are checked during each iteration to see if they have reached a critical 'escape' condition. If that condition is reached, the calculation is stopped, the pixel is drawn, and the next x, y point is examined. For some starting values, escape occurs quickly, after only a small number of iterations. For other starting values it may take hundreds or thousands of iterations to escape. For values within the Mandelbrot set, escape will never occur. The programmer or user must choose how much iteration, or 'depth', they wish to examine. The higher the maximum number of iterations, the more detail and subtlety emerge in the final image, but the longer time it will take to calculate the picture.

The color of each point represents how quickly the values reached the escape point. Often black is used to show values that fail to escape before the iteration limit, and gradually brighter colours are used for points that escape. This gives a visual representation of how many cycles were required before reaching the escape condition.

For Programmers

The definition of the Mandelbrot set, together with its basic properties, suggests a simple algorithm for drawing a picture of the Mandelbrot set. The region of the complex plane we are considering is subdivided into a certain number of pixels. To color any such pixel, let c be the midpoint of that pixel. We now iterate the critical value c under f_c, checking at each step whether the orbit point has modulus larger than 2.

If this is the case, we know that the midpoint does not belong to the Mandelbrot set, and we color our pixel. (Either we color it white to get the simple mathematical image or color it according to the number of iterations used to get the well-known colorful images). Otherwise, we keep iterating for a certain (large, but fixed) number of steps, after which we decide that our parameter is "probably" in the Mandelbrot set, or at least very close to it, and color the pixel black.

In pseudocode, this algorithm would look as follows.

```
For each pixel on the screen do:
{
  x = x0 = x co-ordinate of pixel
  y = y0 = y co-ordinate of pixel

  x2 = x*x
  y2 = y*y

  iteration = 0
  maxiteration = 1000

  while ( x2 + y2 < (2*2)  AND  iteration < maxiteration )
  {

    y = 2*x*y + y0
    x = x2 - y2 + x0

    x2 = x*x
    y2 = y*y

    iteration = iteration + 1
  }

  if ( iteration == maxiteration )
    colour = black
  else
    colour = iteration
}
```

where, relating the pseudocode to c, z and f_c:

- $z = x + iy$
- $z^2 = x^2 + i2xy - y^2$
- $c = x0 + iy0$

and so, as can be seen in the pseudocode in the computation of x and y:

- $x = Re(z^2 + c) = x^2 - y^2 + x0$ and y = $Im(z^2 + c) = 2xy + y0$

The code above has some speed optimizations. The use of the variables x2 and y2 means we save several operations. It saves us two multiplications since then we don't need to do "x*x" and "y*y" in the while-expression. Also note that we put "y = 2*x*y + y0" before "x = x2 - y2 + x0", otherwise we would need to store away "x" in a temporary variable before calculating "y = 2*x*y + y0". The reason we then don't need to store "y" in a temporary variable is that "x=" now doesn't do "y*y" but instead uses the variable "y2".

Continuous (smooth) colouring

Instead of the separate bands of colour produced by the basic algorithm, a more aesthetically-pleasing smoothly-coloured image can be achieved by using a suitable formula to convert the discrete iteration count into a fractional (continuous) number whenever the iteration process 'escapes'. One such formula is the "renormalized" fraction iteration count, an overview of which is provided here (http://linas.org/art-gallery/escape/smooth.html) , with a more detailed explanation and sample images provided here (http://linas.org/art-gallery/escape/escape.html)
. The fractional iteration count can then be converted to a suitable colour; one method is to interpolate between the colours defined for the basic discrete algorithm.

Problems

The Mandelbrot set has some very thin filaments, so even if a given pixel *does*

A portion of the Mandelbrot set centered at (0.282, -0.01)

A portion of the Mandelbrot set.

intersect the Mandelbrot set, it is quite likely that its midpoint will nevertheless escape.

As a result, the algorithm does not behave stably under small perturbations, and will generally not detect small features of the Mandelbrot set. It is precisely this property which led Mandelbrot to conjecture that M is disconnected.

A common way around this problem, which also results in more aesthetically pleasing pictures, is to color any pixel whose midpoint is determined to be an escaping parameter according to the number of iterations it requires to escape. Since parameters closer to the Mandelbrot set will take longer to escape, this method will make the connections between different parts of the Mandelbrot set visible.

Distance estimates

The proof of the connectedness of the Mandelbrot set in fact gives a formula for the uniformizing map of the complement of M
(and the derivative of this map). By the Koebe 1/4-theorem, one can then estimate the distance between the mid-point of our pixel and the Mandelbrot set up to a factor of 4.

In other words, provided that the maximal number of iterations is sufficiently high, one obtains a picture of the Mandelbrot set with the following properties:

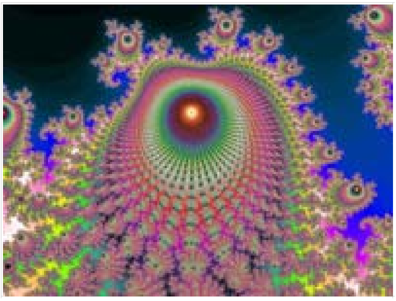
1. Every pixel which contains a point of the Mandelbrot set is colored black.
2. Every pixel which is colored black is close to the Mandelbrot set.

Optimizations

One way to improve calculations is to find out beforehand whether the given point lies within the cardioid or in the period 2 bulb.

To prevent having to do huge numbers of iterations for other points in the set, one can do "periodicity checking"—which means check if a point reached in iterating a pixel has been reached before. If so, the pixel cannot diverge, and must be in the set. This is most relevant for fixed-point calculations, where there is a relatively high chance of such periodicity—a full floating-point (or higher accuracy) implementation would rarely go into such a period.

Art and the Mandelbrot set



"Budding Turbines"

Searching the Mandelbrot set for interesting pictures and transforming them into artistic images often produces remarkably beautiful works. The fractal art article has some examples.

Here
is an animation showing increasing levels of detail (warning, large file). Although it does not start from a view of the whole set, towards the end it clearly shows the recursive nature of the fractal at all scales when a shape similar to the whole comes into view.

The Mandelbrot set in popular culture

St. Louis filmmaker Bill Boll has produced a DVD called 'The Amazing Mandelbrot Set' (http://www.mandelbrotset.net)
containing several high-quality deep zooms of the Mandelbrot set, along with a tutorial.

The Australian band GangGajang (http://www.ganggajang.com/) has a song *Time (and the Mandelbrot set) (http://www.ganggajang.com/lyrics_time.htm)* where the term *Mandelbrot set* is used liberally in the lyrics.

The American singer Jonathan Coulton has a song titled *Mandelbrot Set (http://www.jonathancoulton.com/lyrics/mandelbrot-set)* on his EP *Where Tradition Meets Tomorrow* about the history of the Mandelbrot set, and of Benoît Mandelbrot himself.

Blue Man Group's first album *Audio*
features tracks titled "Mandelgroove", "Opening Mandelbrot", and "Klein Mandelbrot". The album was nominated for a Grammy in 2000. Also, a hidden track on their second album, The Complex, is entitled Mandelbrot No. 4.

The artwork for Canadian band Rush's 16th album, *Test for Echo*, includes the Mandlebrot Set.

The British post-rock band *Mandelbrot Set* released their debut album *All Our Actions Are Constantly Repeated* on October 2006 on Highpoint Lowlife.

Detroit-area melodian Bonedaddy, of Scrummage Records, gives a shout-out to Mandlebrot and his set in the song "Epstein's Face", wherein he states clearly: "Mandlebrot set; 2 legit 2 quit; Epstein's Face is floating through time and space!" In the music video for this song, an image of the Mandlebrot fractal enlarges in front of the viewer's eyes.

A Mandelbrot poster is visible in Eric's room in the That '70s Show sitcom (3rd season), which is an anachronism since those posters only appeared in the 80s.

In the opening credits of *Casino Royale* a plume of gunsmoke coming from a pistol is stylistically represented as clubs (the playing cards suit) repeated as a Mandlebrot pattern.

References

1. ^ Robert Brooks and Peter Matelski, *The dynamics of 2-generator subgroups of PSL(2,C)*, in "Riemann Surfaces and Related Topics", ed. Kra and Maskit, Ann. Math. Stud. 97, 65–71, ISBN 0-691-08264-2
2. ^ Benoît Mandelbrot, *Fractal aspects of the iteration of* $z \mapsto \lambda z(1-z)$ *for complex* λ, z, Annals NY Acad. Sci. **357**, 249/259
3. ^ Adrien Douady and John H. Hubbard, *Etude dynamique des polynômes complexes*, Prépublications mathémathiques d'Orsay 2/4 (1984 / 1985)
4. ^ Mitsuhiro Shishikura, *The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets*, Ann. of Math. **147** (1998) p. 225-267. (First appeared in 1991 as a Stony Brook IMS Preprint (http://www.math.sunysb.edu/preprints.html) , available as arXiv:math.DS/9201282 (http://www.arxiv.org/abs/math.DS/9201282) .)

Further reading

- Lennart Carleson and Theodore W. Gamelin, *Complex Dynamics*, Springer 1993, ISBN 0-387-97942-5
- John W. Milnor, *Dynamics in One Complex Variable* (Third Edition), Annals of Mathematics Studies 160, Princeton University Press 2006, ISBN 0-691-12488-4 (First appeared in 1990 as a Stony Brook IMS Preprint (http://www.math.sunysb.edu/preprints.html) , available as arXiV:math.DS/9201272 (http://www.arxiv.org/abs/math.DS/9201272) .)
- Nigel Lesmoir-Gordon. "The Colours of Infinity: The Beauty, The Power and the Sense of Fractals." ISBN 1-904555-05-5 (The book comes with a related DVD of the Arthur C. Clarke documentary introduction to the fractal concept and the Mandelbrot set.

See also

- Udo of Aachen, a fictional monk, who supposedly discovered the Mandelbrot set in the thirteenth century in an April Fool's hoax perpetrated by British technical writer Ray Girvan
- External ray
- Fractint - an open source fractal generator
- Julia Set

External links

- The Encyclopedia of the Mandelbrot Set (http://www.mrob.com/pub/muency.html) by Robert P. Munafo
- The Mandelbrot and Julia sets Anatomy (http://ibiblio.org/e-notes/MSet/Contents.htm) by Evgeny Demidov
- papers of G. Pastor and M. Romera (http://www.iec.csic.es/~gerardo/Gframe.htm)
- FAQ on the Mandelbrot set (http://www.faqs.org/faqs/fractal-faq/section-6.html)
- PI and the Mandelbrot set (http://home.comcast.net/~davejanelle/mandel.html) , A very interesting connection between PI and the Mandelbrot set, by Dave Boll
- Chaos and Fractals (http://dmoz.org/Science/Math/Chaos_and_Fractals/) at the Open Directory Project (suggest site (http://dmoz.org/cgi-bin/add.cgi?where=Science/Math/Chaos_and_Fractals/))

Julia set

From Wikipedia, the free encyclopedia

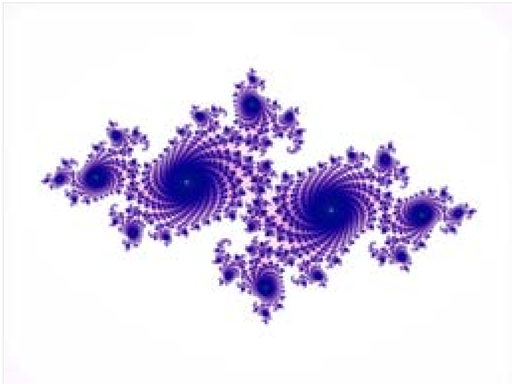
In complex dynamics, the **Julia set** $J(f)$ of a holomorphic function f informally consists of those points whose long-time behavior under repeated iteration of f can change drastically under arbitrarily small perturbations.

The **Fatou set** $F(f)$ of f is the complement of the Julia set: that is, the set of points which exhibit 'stable' behavior.

Thus on $F(f)$, the behavior of f is 'regular', while on $J(f)$, it is 'chaotic'.

These sets are named in honor of the French mathematicians Gaston Julia and Pierre Fatou, who initiated the theory of complex dynamics in the early 20th century.

Contents



A Julia set

Formal definition

3D slice of a 4D Quaternion Julia set

Let

$$f : X \to X$$

be an analytic self-map of a Riemann surface X. We will assume that X is either the Riemann sphere, the complex plane, or the once-punctured complex plane, as the other cases do not give rise to interesting dynamics. (Such maps are completely classified.)

We will be considering f as a discrete dynamical system on the phase space X, so we are interested in the behavior of the iterates f^n of f (that is, the n-fold compositions of f with itself).

The Fatou set of f consists of all points $z \in X$ such that the family of iterates

$$(f^n)_{n \in \mathbb{N}}$$

forms a normal family in the sense of Montel when restricted to some open neighborhood of z.

The Julia set of f is the complement of the Fatou set in X.

Equivalent descriptions of the Julia set

- $J(f)$ is the smallest closed set containing at least three points which is completely invariant under f.
- $J(f)$ is the closure of the set of repelling periodic points.
- For all but at most two points $z \in X$, the Julia set is the set of limit points of the full backwards orbit $\bigcup_n f^{-n}(z)$. (This suggests a simple algorithm for plotting Julia sets, see below.)
- If f is an entire function - in particular, when f is a polynomial, then $J(f)$ is the boundary of the set of points which converge to infinity under iteration.
- If f is a polynomial, then $J(f)$ is the boundary of the filled Julia set; that is, those points whose orbits under f remain bounded.

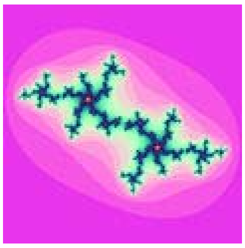
Quadratic polynomials

A very popular complex dynamical system is given by the family of quadratic polynomials,

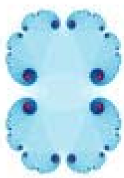
$$f_c(z) = z^2 + c$$

(where c is a complex parameter).

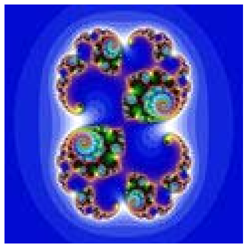
Filled Julia set for f_c, c=φ−2



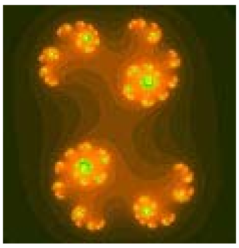
Julia set for f_c, c=(φ−2)+(phi−1)i



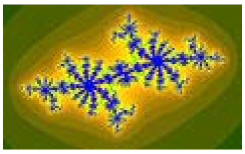
Julia set for f_c, c=0.285



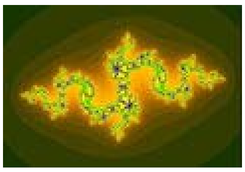
Julia set for f_c, c = 0.285 + 0.01i



Julia set for c=0.45 - 0.1428i



Julia set for f_c, c=-0.70176 -0.3842i

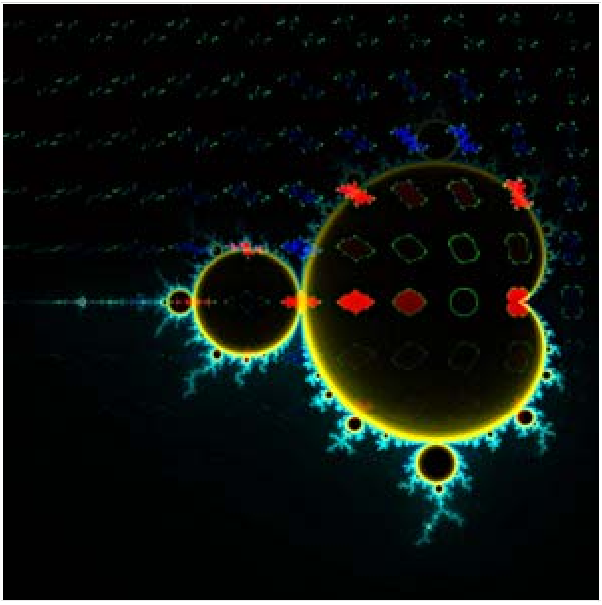


Julia set for f_c, c=-0.835-0.2321i

The parameter plane of quadratic polynomials - that is, the plane of possible *c*-values - gives rise to the famous Mandelbrot set. Indeed, the Mandelbrot set is defined as the set of all *c* such that $J(f_c)$ is connected. For parameters outside the Mandelbrot set, the Julia set is a Cantor set: in this case it is sometimes referred to as **Fatou dust**.

In many cases, the Julia set of *c* looks like the Mandelbrot set in sufficiently small neighborhoods of *c*. This is true, in particular, for so-called 'Misiurewicz' parameters, i.e. parameters *c* for which the critical point is pre-periodic. For instance:

- At *c*= i, the shorter, front toe of the forefoot, the Julia set looks like a branched lightning bolt.
- At *c* = −2, the tip of the long spiky tail, the Julia set is a straight line segment.



Map of 121 Julia sets in position over the Mandelbrot set

Generalizations

The definition of Julia and Fatou sets easily carries over to the case of certain maps whose image contains their domain; most notably transcendental meromorphic functions and Epstein's 'finite-type maps'.

Julia sets are also commonly defined in the study of dynamics in several complex variables.

Plotting the Julia set using backwards iteration

A Julia set plot, generated using backwards iteration

As mentioned above, the Julia set can be found as the set of limit points of the set of pre-images of (essentially) any given point. So we can try to plot the Julia set of a given function as follows. Start with any point z we know to be in the Julia set, such as a repelling periodic point, and compute all pre-images of z under some high iterate f^n of f.

Unfortunately, as the number of iterated pre-images grows exponentially, this is not computationally feasible. However, we can adjust this method, in a similar way as the "random game" method for iterated function systems. That is, in each step, we choose at random one of the inverse images of f.

For example, for the quadratic polynomial f_c, the backwards iteration is described by

$$z_{n+1}^2 = z_n - c.$$

At each step, one of the two square roots is selected at random.

Note that certain parts of the Julia set are quite hard to reach with the reverse Julia algorithm. For this reason, other methods usually produce better images.

See also

- Limit set
- Stable and unstable sets
- No wandering domain theorem

References

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- Adrien Douady and John H. Hubbard, "Etude dynamique des polynômes complexes", *Prépublications mathémathiques d'Orsay* 2/4 (1984 / 1985)
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- Alexander Bogomolny, "Mandelbrot Set and Indexing of Julia Sets (http://www.cut-the-knot.org/Curriculum/Algebra/JuliaIndexing.shtml) " at *cut-the-knot*.
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Fractal dimension

From Wikipedia, the free encyclopedia

In fractal geometry, the **fractal dimension** is a statistical quantity that gives an indication of how completely a fractal appears to fill space, as one zooms down to finer and finer scales. There are many specific definitions of fractal dimensions, none of them should be treated as the universal one. From the theoretical point of view the most important are the Hausdorff dimension, the packing dimension and, more generally, the Rényi dimensions. On the other hand the box-counting dimension is widely used in applications. Although for some classical fractals all these dimensions do coincide, in general they are not equivalent. For example, what is the dimension of the Koch snowflake? It has topological dimension one, but it is by no means a curve-- the length of the curve between any two points on it is infinite. No small piece of it is line-like, but neither is it like a piece of the plane or any other. In some sense, we could say that it is too big to be thought of as a one-dimensional object, but too thin to be a two-dimensional object. Maybe its dimension should be a number between one and two.

Contents

The Many Definitions

There are two main approaches to generate a fractal structure. One is growing from a unit object, and the other is to construct the subsequent divisions of an original structure, like the Sierpinski triangle (Fig.(2))[2]. Here we follow the second approach to define the dimension of fractal structures.

If we take an object with linear size equal to 1 residing in Euclidean dimension, and reduce its linear size to be *l* in each spatial direction, we get *N*(*l*) number of self similar objects within the original object(Fig.(1)). However, the dimension defined by

$$D = \frac{\log N(l)}{\log \frac{1}{l}}.$$

is still equal to its topological or Euclidean dimension[1]. By applying the above equation to fractal structure, we can get the dimension of fractal structure (which is more or less the Hausdorff dimension) as a rational number as expected.

$$D = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \frac{1}{\epsilon}}$$

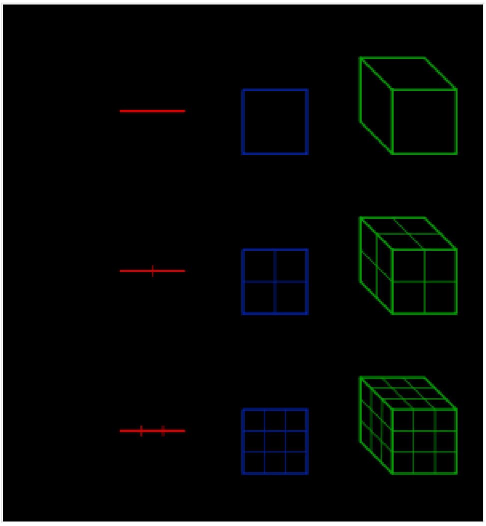

Fig.(1) Another way to define dimension[1]

where *N(ε)* is the number of self-similar structures of linear size ε needed to cover the whole structure.

For instance, the fractal dimension of Sierpinski triangle (Fig.(2)) is given by,

$$D = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \left(\frac{1}{\epsilon}\right)} = \lim_{k \to \infty} \frac{\log 3^k}{\log 2^k} = \frac{\log 3}{\log 2} \approx 1.585$$

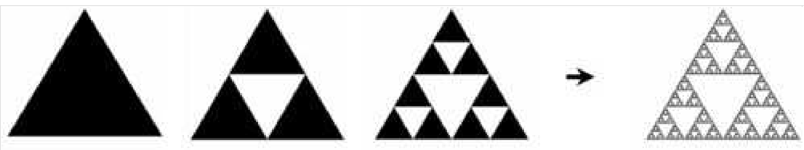

Fig.(2) Sierpinski triangle

Closely related to this is the box-counting dimension, which considers, if the space were divided up into a grid of boxes of size ε, how does the number of boxes scale that would contain part of the attractor? Again,

$$D_0 = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \frac{1}{\epsilon}}$$

Other dimension quantities include the information dimension, which considers how the average information needed to identify an occupied box scales, as the scale of boxes gets smaller:

$$D_1 = \lim_{\epsilon \to 0} \frac{-\langle \log p_\epsilon \rangle}{\log \frac{1}{\epsilon}}$$

and the correlation dimension, which is perhaps easiest to calculate,

$$D_2 = \lim_{\epsilon \to 0, M \to \infty} \frac{g_\epsilon / M^2}{\log \frac{1}{\epsilon}}$$

where M is the number of points used to generate a representation of the fractal or attractor, and g_ε is the number of pairs of points closer than ε to each other.

Rényi dimensions

The last three can all be seen as special cases of a continuous spectrum of generalised or Rényi dimensions of order α, defined by

$$D_\alpha = \lim_{\epsilon \to 0} \frac{\frac{1}{1-\alpha} \log(\sum_i p_i^\alpha)}{\log \frac{1}{\epsilon}}$$

where the numerator in the limit is the Rényi entropy of order α. The Rényi dimension with α=0 treats all parts of the support of the attractor equally; but for larger values of α increasing weight in the calculation is given to the parts of the attractor which are visited most frequently.

An attractor for which the Rényi dimensions are not all equal is said to be a multifractal, or to exhibit multifractal structure. This is a signature that different scaling behaviour is occurring in different parts of the attractor.

Practical dimension estimates are very sensitive to numerical or experimental noise, and particularly sensitive to limitations on the amount of data. Claims based on fractal dimension estimates, particularly claims of low-dimensional dynamical behaviour, should always be taken with a handful of salt — there is an inevitable ceiling, unless *very* large numbers of data points are presented.

See also

- Multifractal analysis
- Fractal
- List of fractals by Hausdorff dimension

Reference

1. ^ ***a b*** http://www.vanderbilt.edu/AnS/psychology/cogsci/chaos/workshop/Fractals.html
2. ^ Fluctuations and Scaling in Biology. Edited by T. Vicsek, 2001