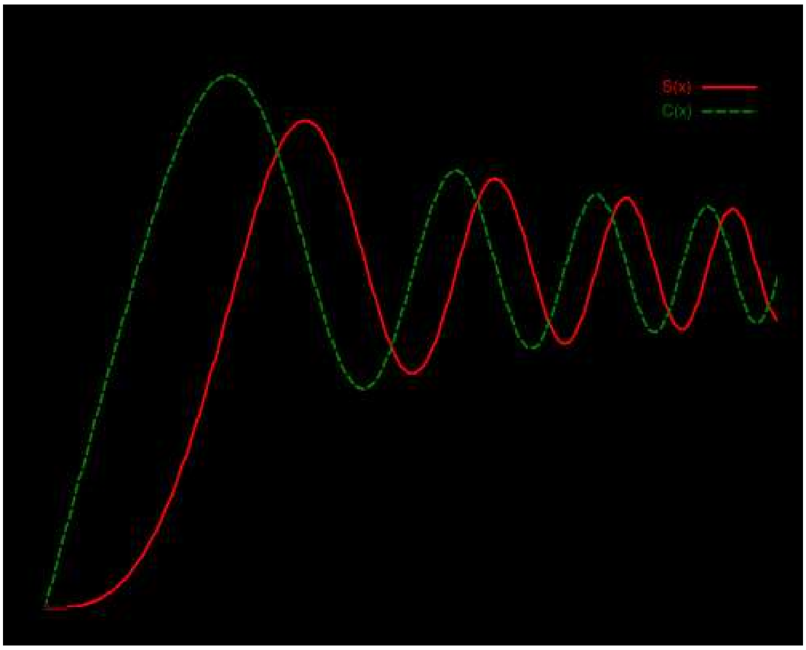
Fresnel integral

From Wikipedia, the free encyclopedia

In mathematics and optics, the two **Fresnel integrals**, $S(x)$ and $C(x)$, arise in the description of near field Fresnel diffraction phenomena, and are the integrals defined as follows:

$$S(x) = \int_0^x \sin(t^2)\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!}$$
$$C(x) = \int_0^x \cos(t^2)\,dt = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!}.$$

Some (including Abramowitz and Stegun, eqs 7.3.1 – 7.3.2) may use $\pi t^2/2$ instead of t^2, in which case the $S(x)$ and $C(x)$ above should be multiplied by $\sqrt{\frac{2}{\pi}}$.

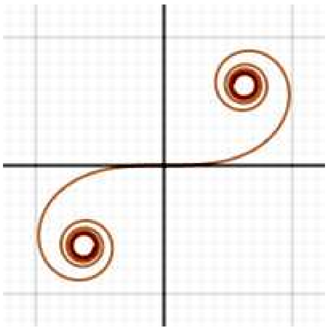


$S(x)$ and $C(x)$ - Note that $C(x)$ does not actually reach 1, as it may appear in the image. The maximum of $C(x)$ is actually about 0.977451424. If $\pi t^2/2$ were used instead of t^2, then the image would be scaled vertically by the factor mentioned above.

Contents

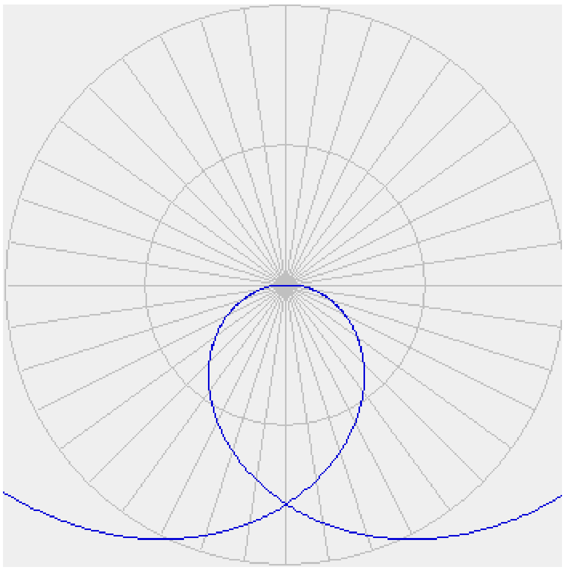
Cornu spiral

The **Cornu spiral**, also known as **clothoid**, is the curve generated by a parametric plot of $S(x)$ against $C(x)$. The Cornu spiral was created by Marie Alfred Cornu as a nomogram for diffraction computations in science and engineering. It is a logical shape with a varying radius, in use for the transition of a straight to a circle curve in roads and railways because a vehicle following the curve at constant speed will have a constant change of rotational acceleration, reducing lateral stress on the rail tracks. However, it may not be the ideal transition spiral, especially at higher speeds, due to other forces acting upon the passengers.



{$C(x)$, $S(x)$} (Note that the spiral should actually converge on the centre of the holes in the image as x tends to positive or negative infinity.)

Following the curve, the length of the curve from {$S(0)$, $C(0)$} to {$S(x)$, $C(x)$} must be equal to x, since $S'(x)^2 + C'(x)^2 = 1$. The total length of the curve (from $x = -\infty$ to ∞) is therefore infinite.

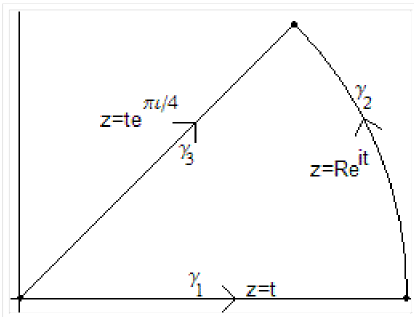


Parametric graph of clothoid loop often found in roller coasters. f(r)=r-pi

Error function

In the domain of complex numbers, the Fresnel integrals can be written using the error function as follows:

$$S(x) = \frac{i\sqrt{\pi}}{4}\left(\operatorname{erf}(\sqrt{i}\,x) - \operatorname{erf}(\sqrt{-i}\,x)\right)$$
$$C(x) = \frac{\sqrt{\pi}}{4}\left(\operatorname{erf}(\sqrt{i}\,x) + \operatorname{erf}(\sqrt{-i}\,x)\right).$$



The "pizza-slice" contour used to calculate the limits of the Fresnel integrals

It is possible (but not trivial) to evaluate the Fresnel integrals in the limits, we have

$$\int_0^\infty \cos t^2\,dt = \int_0^\infty \sin t^2\,dt = \frac{\sqrt{2\pi}}{4} = \sqrt{\frac{\pi}{8}}$$

This can be seen by integrating the function

$$e^{-\frac{1}{2}t^2}$$

around a pizza-slice shaped area beginning in the point (0, 0) (on the complex plane), then going out to (R, 0), up along the arch of the circle centered in (0, 0) and with radius R to the point $Re^{i\pi/4}$ and back to (0, 0) in a straight line.

As R goes to infinity, the integral around the line segment on the edge of the circle will tend to 0, the one along the real axis will tend to the well known Gaussian integral

$$\int_0^\infty e^{-\frac{1}{2}t^2}dt = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}}$$

And the last — along the slope — will evaluate to the Fresnel integrals after some rearrangement.

See also

- Augustin-Jean Fresnel
- Fresnel zone
- Zone plate

References

- Eric W. Weisstein, *Fresnel Integrals (http://mathworld.wolfram.com/FresnelIntegrals.html)* at MathWorld.
- Eric W. Weisstein, *Cornu Spiral (http://mathworld.wolfram.com/CornuSpiral.html)* at MathWorld.
- R. Nave, The Cornu spiral (http://hyperphysics.phy-astr.gsu.edu/hbase/phyopt/cornu.html#c1) , *Hyperphysics* (2002) *(Uses $\pi t^2/2$ instead of t^2.)*
- Milton Abramowitz and Irene A. Stegun, eds. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* New York: Dover, 1972. *(See Chapter 7) (http://www.math.sfu.ca/~cbm/aands/page_297.htm)*

—

Categories: Spirals | Optics | Special functions | Special hypergeometric functions

---